THE ASTROPHYSICAL JOURNAL, 478:L103–L106, 1997 April 1


# THE MIXING LENGTH IN CONVECTIVE STELLAR ENVELOPES IS PROPORTIONAL TO DISTANCE FROM THE CONVECTIVE BOUNDARY

RICHARD B. STOTHERS AND CHAO-WEN CHIN
Institute for Space Studies, NASA/Goddard Space Flight Center, 2880 Broadway, New York, NY 10025
*Received 1996 October 28; accepted 1997 January 15*

## ABSTRACT

A calibration of two different stellar convection theories is made by theoretically reproducing with stellar models the effective temperatures of very luminous red supergiants. This calibration is then compared with previous calibrations using less luminous objects. The corresponding range of stellar masses is 1–20 $M_\odot$. For consistency with previous work, the calibrating parameter is taken to be the convective mixing length $l$. If $l$ is assumed to be proportional to the local pressure scale height $H_p$, the constant of proportionality $\alpha_P$ must vary significantly with stellar mass. If, however, $l$ is assumed to be proportional to the distance $z$ below the outer boundary of the convection zone, the constant of proportionality $\alpha_z$ emerges as a universal constant, within the uncertainties due to possible errors of the observed effective temperatures and of the theoretically calculated low-temperature opacities. In particular, standard mixing-length theory yields a constant $\alpha_z = 2$, whereas the new full-spectrum-of-turbulence theory of Canuto & Mazzitelli yields a constant $\alpha_z = 1$. Physical constraints, laboratory experiments, and observations of turbulent convection in the Earth's atmosphere indicate $\alpha_z \leq 1$. This basic consistency of results over such an enormous range of physical dimensions suggests a very great degree of generality.

*Subject headings:* convection — open clusters and associations: general — stars: interiors — stars: late-type — supergiants — turbulence

## 1. INTRODUCTION

Most theories of stellar envelope convection assume incompressible flow and so cannot provide a characteristic unit of length. How, then, should one express and evaluate the mixing length of a typical turbulent eddy? This question is usually answered by making a further assumption: that the mixing length, $l$, is proportional to the local pressure scale height, $H_P$, or to the distance below the top of the convection zone, $z$, or to some other related distance. Then, the problem of finding the mixing length reduces to finding the value of the appropriate proportionality constant, $\alpha$.

Deep in the convection zone, convective transport of energy is close to being adiabatic. Except for the relatively minor question of convective overshoot at the upper and lower boundaries of the convection zone, a theory of convection (and hence a mixing length) is needed primarily to calculate the temperature gradient in the strongly superadiabatic layers below the radiative atmosphere. In this transition region, stellar structure theory gives $H_P \sim z$ (Stothers & Chin 1995), while convection theory suggests $l \sim H_P$ (Hossain & Mullan 1990) or $l \sim z$ (Canuto & Mazzitelli 1992), which accounts for the fact that $\alpha$ is always semiempirically found to be of order unity. Astrophysical evaluation of the merits of a stellar convection theory, therefore, requires a rather precise evaluation of $\alpha$ for cool stars, preferably over a wide range of masses and metallicities.

In the present Letter, we focus on two very different theories of stellar envelope convection and find that if, on the one hand, the mixing length is cast in the form $l = \alpha_z z$, the value of $\alpha_z$ turns out to be a universal constant, though different for each convection theory. On the other hand, if $l = \alpha_P H_P$, the quantity $\alpha_P$ turns out to vary significantly from star to star.

## 2. DETERMINATIONS OF $\alpha$

The two stellar convection theories under present consideration are the standard version of the mixing-length theory (MLT) (Böhm-Vitense 1958; Cox & Giuli 1968) and a new convection theory that incorporates the full spectrum of turbulent eddy sizes (FST) (Canuto & Mazzitelli 1991, 1992). Recently, Canuto (1996) has shown rigorously that the mixing length in both theories refers to the largest eddy. Furthermore, the two theories formally predict convective fluxes that are not trivial multiples of each other, and therefore the fluxes cannot in general be converted one to the other by simply scaling $\alpha$. However, the stellar transition region for which a convection theory is actually needed is relatively narrow, and consequently the convective flux there varies approximately as some simple power of $l$ for all reasonable convection theories (Gough & Weiss 1976; Canuto & Mazzitelli 1991). In this situation, the convective flux does in fact scale with $\alpha$. Therefore, what we are actually testing is not the correctness or incorrectness of a particular convection theory but rather some formal measure of the convective mixing length. If the empirically derived value of $\alpha$ turns out to be universal, it indicates that the chosen formulation of the mixing length is correct. If $\alpha$ is, in addition, reasonable in magnitude for one of the employed convection theories, then that theory becomes at least a physically viable convection theory.

Tests that we have conducted so far have involved a comparison (at fixed luminosity) of observed and predicted effective temperatures of red giants and red supergiants in the mass range 3–10 $M_\odot$ and in the metallicity range $Z = 0.002$–0.02 (Stothers & Chin 1995, 1996). Such an approach is meaningful because a star's luminosity does not depend on the value of $\alpha$. Metallicity, however, appears to be an undiscriminating probe of convection theories, as $\alpha$ turns out to be not

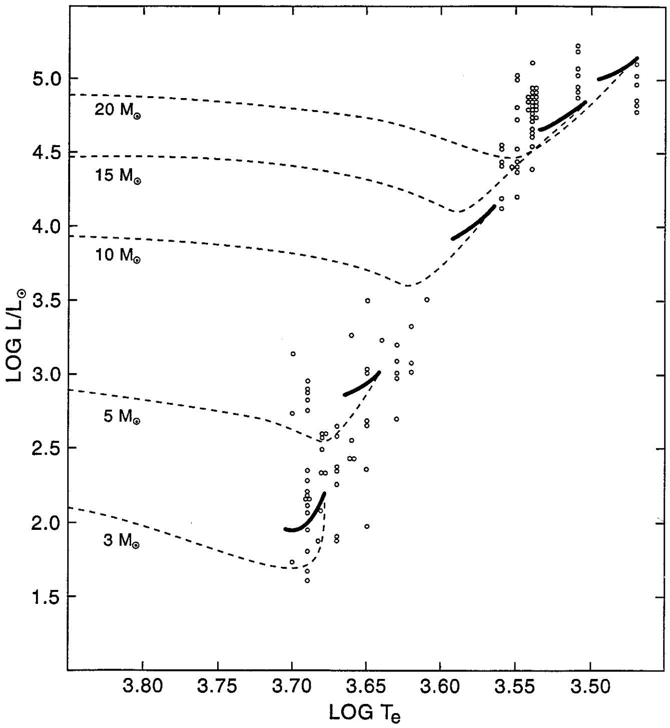


FIG. 1.—H-R diagram showing evolutionary tracks running as far as the second luminosity minimum on the red giant branch. Dashed curves represent very rapid stages. A best-fit value of $\alpha_P$ has been used for each track. Red giants and red supergiants in Galactic open clusters and associations are plotted as open circles.

significantly dependent on metals abundance for either the MLT or the FST model of convection. We also have shown that the slight metallicity dependence of $\alpha$ in the case of low-mass stars both on and off the main sequence, which Chieffi, Straniero, & Salaris (1995) and Salaris & Cassisi (1996) detected with MLT, probably lacks statistical significance, if the typical estimated errors of the observed effective temperatures are taken into consideration. Changing the initial helium abundance makes even less difference, since it causes much smaller horizontal shifts on the H-R diagram from the standard-composition Hayashi line than does changing the metallicity (Hallgren & Cox 1970).

A possible dependence of $\alpha$ on stellar mass (or, equivalently, on stellar luminosity) over the investigated range 3–10 $M_\odot$ (Stothers & Chin 1995) requires more careful evaluation. To extend this comparison further, we present here the results of new full-structure evolutionary calculations for heavier stars of 15 and 20 $M_\odot$ with $Z = 0.02$. All physical input data, including molecular opacities, are the same as in Stothers & Chin (1995). The relevant portions of evolutionary track in the red supergiant region of the H-R diagram are shown in Figure 1. Solid segments indicate the slow (and hence easily observable) stages of central helium depletion, until the star either exits the red region along a blue loop or simply climbs back up the red supergiant branch. Owing to unforeseen computational difficulty, the calculations for 20 $M_\odot$ with the FST model of convection were not continued past the earliest stages of central helium depletion; however, enough evolution was followed to make possible a valid comparison with observational data. The computational difficulty in the FST case, which increases with luminosity, arises from the strong superadiabaticity of the transition region for high luminosities; the enhanced superadiabaticity compared to the MLT case (Canuto & Mazzitelli 1991) causes the structure of the deeper layers to be extremely sensitive to small changes in surface radius and surface luminosity.

In Figure 1 only one track is shown at each stellar mass, because the tracks for the different convection theories and different mixing-length prescriptions nearly overlap if $\alpha$ is chosen so that each track matches the observations at the top of the red supergiant branch. A summary of our main results for 15 and 20 $M_\odot$ is contained in Table 1. Equivalent results for 3, 5, and 10 $M_\odot$ were published earlier.

The observations used here consist of absolute visual magnitudes and MK spectral types of red giants and red supergiants that belong to open clusters and associations in the Galaxy. The open cluster data for the more luminous stars come from Table 6 of Stothers (1991), to which we have added data for the red supergiants in NGC 457 and NGC 2439 from Table 6 of Harris (1976), while the association data come from Table 8 of Humphreys (1978). Bolometric corrections and effective temperatures are taken from Lee (1970). Lee's effective temperatures are virtually identical, for the same $V - K$ color and MK spectral type, to those of Johnson (1966) and of Di Benedetto (1993). Minor differences among the published bolometric corrections (Johnson 1966; Lee 1970; Elias, Frogel, & Humphreys 1985) have little consequence for our present purposes, since they affect, to a slight extent, only the luminosities. The observed stars are plotted in Figure 1, together with our previously discussed data for red giants of lower luminosity. Metallicities of the stars range around the solar value, and ages extend from $1 \times 10^7$ yr to $4 \times 10^8$ yr, yet a mean Hayashi line is readily definable and can be compared with the stellar models.

Consolidating all our results for red giants and red super-

TABLE 1

THEORETICAL RED SUPERGIANT BRANCHES

| $M/M_\odot$ | Convection Theory | $l$ | Deepest $q_{env}$ | Red Top log($L/L_\odot$) | Red Top log $T_e$ | Second Red Bottom log($L/L_\odot$) | Second Red Bottom log $T_e$ |
|---|---|---|---|---|---|---|---|
| 15............ | MLT | $1.5H_P$ | 0.285 | 4.86 | 3.50 | 4.66 | 3.53 |
| | MLT | $2.0z$ | 0.279 | 4.74 | 3.50 | 4.54 | 3.54 |
| | FST | $1.0z$ | 0.278 | 4.74 | 3.49 | 4.54 | 3.52 |
| 20............ | MLT | $1.3H_P$ | 0.347 | 5.16 | 3.47 | 5.01 | 3.49 |
| | MLT | $2.0z$ | 0.327 | 5.09 | 3.47 | 4.92 | 3.50 |
| | FST | $1.0z$ | 0.320 | 5.10 | 3.46 | ... | ... |

NOTE.—$q_{env}$ is the stellar mass fraction at the base of the outer convection zone.

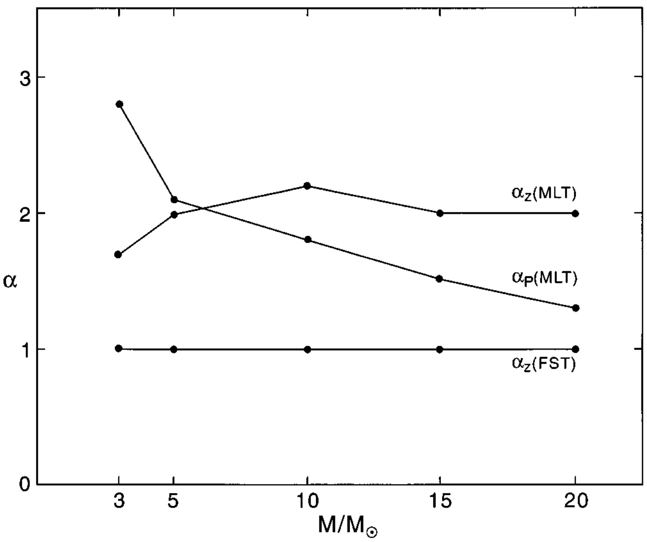


FIG. 2.—Mixing-length ratio, $\alpha$, versus stellar mass, as inferred from Galactic red giants and red supergiants. The 2 $\sigma$ uncertainty in $\alpha$ is $\sim$15%.

giants in Figure 2, we verify that $\alpha_P$ (MLT) declines significantly with increasing stellar mass, from a value of 2.8 at 3 $M_\odot$ to 1.3 at 20 $M_\odot$. The 2 $\sigma$ estimated error in all the $\alpha$ values is $\sim$15%. For the Sun, $\alpha_P$ (MLT) $\approx$ 2.0 if similar input physics are used (Kim, Demarque, & Guenther 1991; Guenther et al. 1992; Lydon, Fox, & Sofia 1993; Sackmann, Boothroyd, & Kraemer 1993).

In our earlier study, we surmised that the apparent rise of $\alpha_z$ (MLT) from 1.7 at 3 $M_\odot$ to 2.2 at 10 $M_\odot$ was possibly significant. But our new results for 15 and 20 $M_\odot$, both giving $\alpha_z$ (MLT) = 2.0, indicate that $\alpha_z$ (MLT) is actually constant within the $\sim$15% estimated error. The origin of the small deviations from the mean can be attributed mostly to uncertainty of the theoretical and observational matchups on the H-R diagram, owing in part to cosmic scatter. For the Sun, $\alpha_z$ (MLT) $\approx$ 2.0, as obtained by interpolation among the solar models of Canuto & Mazzitelli (1991, 1992).

In the case of the FST model of convection, we find $\alpha_z$ (FST) = 1.0 from our analysis of red giants and red supergiants of 3–20 $M_\odot$. For the Sun, Canuto & Mazzitelli (1991, 1992) similarly found $\alpha_z$ (FST) $\approx$ 1.0. The same result has been derived from analyses of $\alpha$ Centauri A and B, other low-mass main-sequence stars, and red giants in the old open cluster M67 and in various globular clusters (D'Antona, Mazzitelli, & Gratton 1992; D'Antona & Mazzitelli 1994; Mazzitelli, D'Antona, & Caloi 1995; Fernandes & Neuforge 1995).

"Hot bottom burning" in models of luminous asymptotic giant branch stars of 5–7 $M_\odot$ occurs, as observationally required, if $\alpha_z$ (FST) = 1.0 or if $\alpha_P$ (MLT) $\geq$ 2.5 (D'Antona & Mazzitelli 1996). The blue edge of the DB white dwarf instability strip in the H-R diagram, too, is compatible with either $\alpha_z$ (FST) = 1.0 or $\alpha_P$ (MLT) > 1 (Althaus & Benvenuto 1996). The clear inconsistency of the $\alpha_P$ prescription seems to be present among many classes of stars, whereas the $\alpha_z$ prescription works very successfully.

## 3. POSSIBLE ERRORS

Several potential sources of error that could affect the derived values of $\alpha_P$ and $\alpha_z$ need to be evaluated. Errors inevitably creep into the stellar models, the observations, the transformations from observational to theoretical quantities, and the matchups of theoretical models to observed stars. Our earlier discussion of the mass range 3–10 $M_\odot$ suggested a 2 $\sigma$ error of 15% for all the derived $\alpha$ values (Stothers & Chin 1995).

Here we concentrate on possible errors occurring at 20 $M_\odot$ due to the most important of the known uncertainties. First, our neglect of stellar wind mass loss might be thought important, but it is not, because the theoretical and observational matchups in the H-R diagram are done at the coolest observed effective temperature, which the star attains very quickly on its first ascent of the red supergiant branch before much mass can be lost. Second, since the coolest effective temperature at 20 $M_\odot$ is so low, the fitted opacity formula that we have used for the outer layers of the stellar models (Stothers & Chin 1993) has had to be substantially extrapolated in order to calculate the structure of the atmosphere. When we replaced the formula with the full opacity tables, the top of the red supergiant branch became hotter, but only by 0.02 dex. Third, the latest spectral subtype shown by the most luminous red supergiants is usually M4 (log $T_e$ = 3.47), but three variable red supergiants are known to appear as late as M5 (log $T_e$ = 3.45), at least on occasions (Blanco 1955; Humphreys & Ney 1974). This range of effective temperatures, if treated as a 2 $\sigma$ estimated error, can be combined with the estimated 2 $\sigma$ error incurred by our conversion between mean spectral subtype and effective temperature, which is $\pm$0.01 dex. When we apply the relation $\partial \log T_e / \partial \log \alpha \approx 0.4$ obtained from our 20 $M_\odot$ models, the total estimated 2 $\sigma$ error in $\alpha$ is found to be $\sim$15%.

## 4. CONCLUSION

Over a wide range of stellar masses, and with the use of two very different theories of stellar envelope convection, we find that $l/z$, or $\alpha_z$, is essentially a constant. However, $\alpha_z$ (MLT) = 2.0, whereas $\alpha_z$ (FST) = 1.0. Since the physical size of the convection zone implies that $l$ cannot exceed $z$, the FST model of convection is clearly preferred. Other astrophysical tests of the FST model, which are more heavily weighted by layers closer to the stellar surface, also suggest $\alpha_z \approx 1.0$; these tests involve nonradial pulsation frequencies of the Sun (Games Paternò et al. 1993; Basu & Antia 1994a, 1994b; Baturin & Mironova 1995) and spectral continua and spectral line intensities of the Sun and $\alpha$ Circini (Kupka 1996). It is not known whether $\alpha_z$ (MLT) = 2.0 would work as effectively.

In contrast, the large inferred variation of $\alpha_P$ with stellar mass demonstrates that the usual prescription $l = \alpha_P H_P$ cannot be correct or, at least, that it is not sufficiently self-consistent to be useful in actual practice. Three-dimensional numerical simulations of envelope convection in solar-type stars also suggest the inadequacy of an $\alpha_P$ prescription (Nordlund & Dravins 1990; Lydon, Fox, & Sofia 1992; Kim et al. 1996). When fitted to the analytical MLT equations, the numerical results show that $\alpha_P$ has to increase from zero at the top of the convection zone to some value of order unity deeper in the transition region. Interestingly, this behavior can be approximately mimicked by taking $l$ proportional to $z$. Nevertheless, the simple MLT and FST models cannot hope to capture all aspects of the full numerical simulations.

Laboratory experiments also suggest that $l$ is proportional to distance from the boundary (Davies 1972, p. 18), and so do observations of turbulent convection in the Earth's atmosphere (Priestley 1959, p. 72). On the much grander scale of stellar interiors the same semiempirical relation seems to hold. Canuto & Mazzitelli (1991) and Böhm & Stückl (1967) have given general supporting arguments of a more theoretical nature. In all of these cases, in which the length scale ranges from $10^2$ to $10^{14}$ cm, one finds that observation, experiment, and theory indicate a constant of proportionality, $\alpha_z$, that is close to unity.

We acknowledge useful discussions with V. M. Canuto and helpful suggestions from the referee. This work was supported by the NASA Astrophysics and Climate Research Programs.

## REFERENCES


Althaus, L. G., & Benvenuto, O. G. 1996, MNRAS, 278, 981
Basu, S., & Antia, H. M. 1994a, J. Astrophys. Astron., 15, 143
———. 1994b, MNRAS, 269, 1137
Baturin, V. A., & Mironova, I. V. 1995, Astron. Rep., 39, 105
Blanco, V. M., 1955, ApJ, 122, 434
Böhm, K.-H., & Stückl, E. 1967, Z. Astrophys., 66, 487
Böhm-Vitense, E. 1958, Z. Astrophys., 46, 108
Canuto, V. M. 1996, ApJ, 467, 385
Canuto, V. M., & Mazzitelli, I. 1991, ApJ, 370, 295
———. 1992, ApJ, 389, 724
Chieffi, A., Straniero, O., & Salaris, M. 1995, ApJ, 445, L39
Cox, J. P., & Giuli, R. T. 1968, Principles of Stellar Structure (New York: Gordon & Breach)
D'Antona, F., & Mazzitelli, I. 1994, ApJS, 90, 467
———. 1996, ApJ, 470, 1093
D'Antona, F., Mazzitelli, I., & Gratton, R. G. 1992, A&A, 257, 539
Davies, J. T. 1972, Turbulence Phenomena (New York: Academic)
Di Benedetto, G. P. 1993, A&A, 270, 315
Elias, J. H., Frogel, J. A., & Humphreys, R. M. 1985, ApJS, 57, 91
Fernandes, J., & Neuforge, C. 1995, A&A, 295, 678
Gough, D. O., & Weiss, N. O. 1976, MNRAS, 176, 589
Guenther, D. B., Demarque, P., Kim, Y.-C., & Pinsonneault, M. H. 1992, ApJ, 387, 372
Hallgren, E. L., & Cox, J. P. 1970, ApJ, 162, 933
Harris, G. L. H. 1976, ApJS, 30, 451
Hossain, M., & Mullan, D. J. 1990, ApJ, 354, L33
Humphreys, R. M. 1978, ApJS, 38, 309
Humphreys, R. M., & Ney, E. P. 1974, ApJ, 194, 623
Johnson, H. L. 1966, ARA&A, 4, 193
Kim, Y.-C., Demarque, P., & Guenther, D. B. 1991, ApJ, 378, 407
Kim, Y.-C., Fox, P. A., Demarque, P., & Sofia, S. 1996, ApJ, 461, 499
Kupka, F. 1996, in Stellar Surface Structure, ed. K. G. Strassmeier & J. L. Linsky (Dordrecht: Kluwer), 557
Lee, T. A. 1970, ApJ, 162, 217
Lydon, T. J., Fox, P. A., & Sofia, S. 1992, ApJ, 397, 701
———. 1993, ApJ, 413, 390
Mazzitelli, I., D'Antona, F., & Caloi, V. 1995, A&A, 302, 382
Nordlund, A., & Dravins, D. 1990, A&A, 228, 155
Paternò, L., Ventura, R., Canuto, V. M., & Mazzitelli, I. 1993, ApJ, 402, 733
Priestley, C. H. B. 1959, Turbulent Transfer in the Lower Atmosphere (Chicago: Univ. Chicago Press)
Sackmann, I.-J., Boothroyd, A. I., & Kraemer, K. E. 1993, ApJ, 418, 457
Salaris, M., & Cassisi, S. 1996, A&A, 305, 858
Stothers, R. B. 1991, ApJ, 383, 820
Stothers, R. B., & Chin, C.-w. 1993, ApJ, 412, 294
———. 1995, ApJ, 440, 297
———. 1996, ApJ, 469, 166